

On Minimum Uncertainty States

Pankaj Sharan

Physics Department, Jamia Millia Islamia, New Delhi, 110 025, India

Abstract

Necessary and sufficient condition for the existence of a minimum uncertainty state for an arbitrary pair of observables is given.

Let the states of a physical system be represented by normalized vectors in a Hilbert space \mathcal{H} . For two vectors ϕ and ψ in \mathcal{H} , denote the inner product by (ψ, ϕ) and define the norm $\|\phi\|$ of ϕ by $\|\phi\|^2 = (\phi, \phi)$. Let A and B be two observables; that is, self-adjoint operators. Let the observable C be defined by the commutator $[A, B] = iC$. The expectation value $(\psi, A\psi)$ of A is denoted by a . Similarly, expectation values of B and C in the state ψ are denoted b and c respectively.

The statement of the uncertainty inequality is

$$\Delta A \Delta B \geq \frac{1}{2}|c|, \quad (1)$$

where the variance (or uncertainty) of A in the state ψ is defined as $\Delta A = \|(A - a)\psi\|$ and a similar formula for ΔB . We say that ψ is a minimum uncertainty state (MUS) for the pair A, B if the equality is achieved in (1) above, that is, if

$$\Delta A \Delta B = \frac{1}{2}|c|. \quad (2)$$

The proof of the uncertainty inequality is a direct application of the Schwarz inequality which states that

$$|(\psi, \phi)| \leq \|\psi\| \|\phi\| \quad (3)$$

for any two vectors ϕ and ψ in \mathcal{H} . We assume that one of the vectors (say ϕ) is non-zero to avoid triviality. The Schwarz inequality becomes an equality if and only if ψ can be written as the other (non-zero) vector ϕ multiplied by a complex number z

$$\psi = z\phi. \quad (4)$$

The proof of the uncertainty inequality is as follows. Denote by $\text{Im } z$ the imaginary part of a complex number z . The Schwarz inequality implies

$$\begin{aligned}
\Delta A \Delta B &= \|(A - a)\psi\| \|(B - b)\psi\| \\
&\geq |((A - a)\psi, (B - b)\psi)| && \text{Inequality 1} \\
&\geq |\text{Im}((A - a)\psi, (B - b)\psi)| && \text{Inequality 2} \\
&= \left| \frac{1}{2i} [((A - a)\psi, (B - b)\psi) - ((B - b)\psi, (A - a)\psi)] \right| \\
&= \frac{1}{2} |c|.
\end{aligned}$$

The condition for ψ to be a MUS for A, B is that at both the places above (Inequality 1 and 2) the equality must be satisfied. The first one is satisfied if and only if there is a complex number z such that

$$(A - a)\psi = z(B - b)\psi \quad (5)$$

where we assume $\Delta B = \|(B - b)\psi\| \neq 0$ to avoid the trivial case when both ΔA and ΔB are zero. By taking norm on both sides of the above equation we also note that

$$\Delta A = |z| \Delta B. \quad (6)$$

The second inequality (Inequality 2) becomes an equality if and only if the real part of $((A - a)\psi, (B - b)\psi)$ is zero. This happens if

$$((A - a)\psi, (B - b)\psi) + ((B - b)\psi, (A - a)\psi) = 0$$

which, in the light of $(A - a)\psi = z(B - b)\psi$ implies that $\text{Re } z = 0$. In other words, $z = i\lambda$ for a real number λ . The magnitude of λ follows from (6) above as

$$|\lambda| = \frac{\Delta A}{\Delta B}. \quad (7)$$

To obtain the sign of λ we proceed as follows. Write $z = i\lambda$ in (5) and calculate

$$\|(A - i\lambda B)\psi\|^2 = |a - i\lambda b|^2 = a^2 + \lambda^2 b^2. \quad (8)$$

The left hand side is

$$\|(A - i\lambda B)\psi\|^2 = ((A - i\lambda B)\psi, (A - i\lambda B)\psi) = (\psi, (A + i\lambda B)(A - i\lambda B)\psi),$$

and

$$(A + i\lambda B)(A - i\lambda B) = A^2 + \lambda^2 B^2 + \lambda C.$$

Substituting these in (8) and using $(\Delta A)^2 = (\psi, A^2 \psi) - a^2$, $\Delta A = |\lambda| \Delta B$ etc. we get,

$$2\lambda^2 (\Delta B)^2 + \lambda c = 0$$

which shows that the sign of λ must be opposite to that of c .

With the notation as above, we have proved the following theorem :

For ψ to be a MUS for the pair A, B (with $\Delta B \neq 0$) the necessary and sufficient condition is that

$$(A - a)\psi = i\lambda(B - b)\psi$$

where λ is a real number whose magnitude is given by $|\lambda| = \Delta A / \Delta B$ and whose sign is opposite to that of c .

We see that the condition for MUS can also be written as

$$(A - i\lambda B)\psi = (a - i\lambda b)\psi, \quad (9)$$

which means that ψ must be an eigenvector of the non-hermitian operator $A - i\lambda B$ with the complex eigenvalue $a - i\lambda b$.

A well-known example of MUS is the gaussian wave-packets in one dimension:

$$\psi = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp \left[ikx - \frac{(x - x_0)^2}{4\sigma^2} \right] \quad (10)$$

for the pair of operators $A = x$ and $B = -id/dx$. Here $a = x_0, b = k, \Delta A = \sigma$ and $\Delta B = 1/(2\sigma)$. Thus $|\lambda| = 2\sigma^2$, and because $c = 1 > 0$ we have $\lambda = -2\sigma^2$. One can check that the wave packet above is the eigenfunction of the operator

$$\left(x + 2\sigma^2 \frac{d}{dx} \right)$$

with complex eigenvalue $x_0 + 2i\sigma^2 k$.

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